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DYNAMIC SIMULATION OF CONTROLS IN CERTAIN PARABOLIC SYSTEMS*

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Problems of the dynamic simulation of controls in parabolic systems involving dissipative operators are considered. Algorithms, stable with respect to information noise and computation errors, which reconstruct the unknown controls for a fairly general class of systems, are described. Examples are presented.

A method was proposed previously /1, 2/ for investigating problems of reconstructing the characteristics of dynamic systems, based on ideas of the theory of positional control /3/ and the theory of ill-posed problems /4/ and valid for systems with a finite number of degrees of freedom. This method will now be extended to new classes of systems with distributed parameters.

1. The content of the problem studied here can be illustrated through a model example which describes the propagation of oxygen in absorbent tissue /5/. The absorbent tissue is assumed to occupy a region $\Omega \subset R^n$ with boundary Γ. The concentration of oxygen in the tissue at a time $t \in T = [t_0, \vartheta]$ at a point $\eta \in \Omega$ is denoted by $y(\eta, t)$. During the time T a certain amount of oxygen is absorbed. The rate of absorption $u(\eta, t)$ is unknown, but at discrete times $\tau_i \in T$, $\tau_{i+1} = \tau_i + \delta_i$, $\delta_i > 0$, $i \in [0{:}\ m-1]$, $\tau_0 = t_0$, $\tau_m = \vartheta$ the concentration of oxygen $y(\eta, \tau_i)$ is measured to within a certain accuracy, i.e., a function $\psi(\eta, \tau_i)$ approximating $y(\eta, \tau_i)$ is determined. It is required to indicate a procedure for computing $u(\eta, t)$ synchronously with the absorption process.

A mathematical model of the absorption process may be described by the relations

$$y_t^{\cdot}(\eta, t) - \Delta y(\eta, t) + \partial I_K(y(\eta, t)) \ni u(\eta, t) \tag{1.1}$$
$$t \in T, \quad y(\eta, t_0) = y_0(\eta)$$

Here ∂I_K is the subdifferential of the characteristic function of the set $K = \{w \in L_2(\Omega) \mid w(\eta) \geqslant 0$ for a.e. $\eta \in \Omega\}$. We shall refer to the function $u(\cdot)$ as the control. Our problem is to reconstruct the control.

In accordance with the approach used in /1, 2/, we will now compute the unknown control $u(\cdot)$ as follows. System (1.1) will be associated with a control system M (the model) with a control $v(\cdot)$ and a phase trajectory $z(\cdot)$. We shall then construct an algorithm to shape the control $v(\cdot)$ in the model, based on the feedback principle $v(\cdot) = v(\cdot; y(\cdot), z(\cdot))$, such that $v(\cdot)$ approximates the unknown control in a suitable sense. Consequently, we are replacing the problem of computing the unknown control by that of devising an algorithm to construct the control in the model. This algorithm will essentially be the required algorithm for reconstructing the unknown control. It must be stable with respect to distortion of the input information.

In this paper the problem of reconstructing the control is considered for non-linear parabolic inclusions involving dissipative operators. We will first investigate the problem of reconstructing distributed controls in systems

$$y^{\cdot}(t) \in Ay(t) + Bu(t) + f(t); \quad t \in T, \quad y \in E, \quad y(t_0) = y_0 \tag{1.2}$$

which include (1.1) as a special case. Questions of this type are then considered for boundary control problems described by the relations

$$y^{\cdot} = \sigma y + B_1 u_1, \quad y \in E; \quad \tau y = B_2 u_2, \quad y(t_0) = y_0 \tag{1.3}$$

Essential use will be made here of some ideas from /6, 7/, as well as results from /8/. In the concluding part of the paper the constructions will be illustrated by means of examples.

We shall use the following notation. $L(U, X)$ is the Banach space of continuous linear operators from U to X, $C([a, b]; E)$ and $L_2([a, b]; E)$ are the standard spaces, $W^{1,2}([a, b]; E)$ is the space of strongly absolutely continuous functions with first derivatives in $L_2([a, b]; E)$; $W^{1,2}((a, b]; E)$ is the space of functions $w(\cdot): [a, b] \to E$ such that for any $\varepsilon > 0$ $w(\cdot): [a+\varepsilon, b] \to E$ is an element of the space $W^{1,2}([a+\varepsilon, b]; E)$; Δ is a partition of the interval

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T of the mesh δ, i.e., a set of points $\{\tau_i\}$, $\tau_i < \tau_{i+1}$, $i \in [0: m-1]$, $\tau_0 = t_0$, $\tau_m = \vartheta$, $\delta = \max_i (\tau_{i+1} - \tau_i)$; $\partial\varphi$ is the subdifferential of φ: $E \to (-\infty, +\infty]$; $\partial\varphi^0(y) = \{z \in E \mid |z|_E = \inf |z|_E,\ z \in \partial\varphi(y)\}$; $\bar{A}$ is the closure of a set $A \subset E$; $D(\varphi) = \{y \in E \mid \varphi(y) < +\infty\}$; $\Omega \subset R^n$ is a region with boundary Γ; $Q = \Omega \times (t_0, \vartheta)$; $\Sigma = \Gamma \times (t_0, \vartheta)$.

2. In a real Hilbert space $(E, |\cdot|)$ we consider a control system Σ described by an inclusion relation (1.2) with non-linear multivalent dissipative operator $A = -\partial\varphi$, φ: $E \to (-\infty, +\infty]$ a convex, proper, lower semicontinuous function, $f(\cdot) \in L_2(T; E)$ a given perturbation, $u(t) \in P \subset U$ a control, P a convex, bounded, closed set, $(U, \|\cdot\|)$ a real Hilbert space, $B \in L(U, E)$. Henceforth we shall assume without loss of generality that $\varphi(0) = 0$, $\varphi(y) \geqslant 0$.

A function $y(\cdot) = y(\cdot; y_0, u(\cdot))$ is called a strong solution of system (1.2) for a control $u(\cdot) \in L_2(T; U)$ and initial state y_0 if $y(t_0) = y_0$, $y(\cdot) \in C(T; E) \cap W^{1,2}(T; E)$ and for a.e. $t \in T$ the function satisfies the equation

$$y^{\cdot}(t) = (Ay(t) + Bu(t) + f(t))^0 \tag{2.1}$$

It is known (/9/, Proposition 5) that for any $y_0 \in D(\varphi)$, $u(\cdot) \in L_2(T; U)$, $f(\cdot) \in L_2(T; E)$ there exists a unique strong solution of (1.2), and moreover

$$|y^{\cdot}(\cdot)|_{L_2(T;E)} \leqslant |Bu(\cdot) + f(\cdot)|_{L_2(T;E)} + \varphi^{1/2}(y_0) \tag{2.2}$$

A function $y(\cdot) = y(\cdot; y_0, u(\cdot))$ is called a weak solution of (1.2) if there is a sequence $\{y^{(n)}\} \in D(\varphi)$ such that $y^{(n)} \to y_0$ in E, $y^{(n)}(\cdot) = y(\cdot; y^{(n)}, u(\cdot)) \to y(\cdot)$ in $C(T; E)$. A weak solution $y(\cdot) = y(\cdot; y_0, u(\cdot))$ exists for any $y_0 \in \overline{D(\varphi)}$, $u(\cdot) \in L_2(T; U)$, $f(\cdot) \in L_2(T; E)$ and has the following properties: $y(\cdot) \in C(T; E) \cap W^{1,2}((t_0, \vartheta;] E)$ and for a.e. $t \in T$ the function satisfies (2.1) (/10, Theorem 22), $t \to \varphi(y(t)) \in C((t_0, \vartheta]; R)$ (/9/, Proposition 5).

Let us briefly recall the essence of our problem. A motion $y(\cdot)$ of system (1.2) is a weak solution, generated by an unknown control $u(\cdot), u(t) \in P$ for a.e. $t \in T$. Both $y(\cdot)$ and the initial state y_0 are also unknown. However, one has a continuous flow of information about $y(\cdot)$ - elements $\psi_i \in E$ produced at times $\tau_i \in \Delta$, such that $|\psi_i - y(\tau_i)| \leqslant h$, $h \geqslant 0$. It is required to design an algorithm that reconstructs $u(\cdot)$.

To compute $u(\cdot)$ we introduce a model, described by the inclusion

$$z^{\cdot}(t) \in Az(t) + Bv^h(t) + f(t), \quad t \in T \tag{2.3}$$

with initial condition $z(t_0) = z_0 \in D(\varphi)$, $|z_0 - \psi_0| \leqslant 2h$ and trajectory $z(\cdot) = z(\cdot; z_0, v^h(\cdot))$ — a strong solution of (2.3). We shall control the model according to the feedback principle. Specifically, we stipulate the following procedure for constructing the control $v^h(\cdot) = v^h(\cdot; \psi_i, y(\tau_i))$:

$$v^h(t) = v_i \text{ for a.e. } t \in [\tau_i, \tau_{i+1}) \tag{2.4}$$
$$v_i \in \{v \in P \mid l(s_i, v) \leqslant l(s_i) + h\}$$
$$l(s_i, v) = 2(s_i, Bv)_E + \alpha\|v\|^2 \tag{2.5}$$
$$l(s_i) = \inf\{l(s_i, v) \mid v \in P\}, \quad s_i = z(\tau_i) - \psi_i$$

Let $\omega(\cdot)$ denote the modulus of continuity of the function $y(\cdot)$, i.e.,

$$\omega(\delta) = \sup\{|y(t) - y(\xi)| \mid |t - \xi| \leqslant \delta,\ t, \xi \in T\}$$

and U_* the set of all controls $v(\cdot) \in L_2(T; U)$ satisfying the condition $z(\cdot; y_0, v(\cdot)) = y(\cdot)$, $v(t) \in P$ for a.e. $t \in T$. It can be verified that U_* contains a unique element $u_*(\cdot)$ with minimal norm in the space $L_2(T; U)$.

Suppose that the quantities $\alpha(h) > 0$, $\delta(h) > 0$, $z_0 = z_0^{(h)}$, $\alpha(h) \to 0$, $h \cdot \alpha^{-1}(h) \to 0$, $|z_0 - y_0| \leqslant ch$ have already been chosen and

$$\alpha^{-1}(h)\{\omega(\delta(h)) + \delta(h)(1 + \varphi^{1/2}(z_0))\} \to 0 \quad \text{as} \quad h \to 0+ \tag{2.6}$$

Theorem 2.1.

$$\|v^h(\cdot) - u_*(\cdot)\|_{L_2(T;U)} \to 0 \quad \text{as} \quad h \to 0+$$

The proof follows the same lines as in /1/ and is based on the fact that the algorithm for generating the control $v^h(\cdot)$ in the model guarantees "stabilization" in time T of the functional

$$\varepsilon_\alpha(t) = |z(t) - y(t)|^2 + \alpha\int_{t_0}^{t}\{\|v^h(\xi)\|^2 - \|u_*(\xi)\|^2\}\,d\xi$$

Remarks. 1. Theorem 2.1 is also true if (2.6) is replaced by the weaker condition

$$\alpha^{-1}(h)\Big\{\sum_{i=0}^{m-1}\delta_i\omega_i+\delta(1+\varphi^{1/2}(z_0^{(h)}))\Big\}\to 0 \text{ as } h\to 0+, \tag{2.7}$$
$$\delta_i=\tau_{i+1}-\tau_i,\quad \omega_i=\sup\{|y(t)-y(\tau_i)|\mid t\in[\tau_i,\tau_{i+1}]\}$$

2. If $y_0\in D(\varphi)$, we can assume that $z_0=y_0$. By (2.2), condition (2.7) will then hold if $\delta\alpha^{-1}(h)\to 0$ as $h\to 0+$.

3. When generating the control $v^h(\cdot)$ one can replace $z(\tau_i)$ by an approximation $z^*(\tau_i)$: $|z(\tau_i)-z^*(\tau_i)|\leqslant ch$.

3. We will now consider the case of system (1.3). Let us assume that E, X, U_1, U_2 are real Hilbert spaces, $\sigma: E\to E$ a closed linear operator with dense domain, $\tau: E\to X$ a linear (boundary) operator, $B_1: U_1\to E$ and $B_2: U_2\to X$ continuous linear operators, $u_1=u_1(t)\in P_1$, $u_2=u_2(t)\in P_2$ for a.e. $t\in[0,\vartheta]$ $(t_0=0)$, $P_1\subset U_1$ and $P_2\subset U_2$ convex, bounded, closed sets.

Define an operator $A\in L(E,E)$ as follows:

$$D(A)=\{y\in D(\sigma)\mid \tau y=0\},\quad Ay=\sigma y,\quad \forall y\in D(A)$$

In the sequel we shall make the following assumptions /8/.

1. $D(\sigma)\subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous.

2. A is the infinitesimal generator of a strongly continuous contracting semigroup $\{S(t)\mid t\geqslant 0\}$ on E.

3. There exists an operator $B\in L(U_2,E)$ with the properties

$$\sigma B\in L(U_2,E),\quad \tau(Bu)=B_2u,\quad \forall u\in U_2$$
$$|Bu|\leqslant c\,|B_2u|_X,\quad \forall u\in U_2$$

where c is a positive constant.

4. For any $t\in(0,\vartheta]$ and $u\in U_2$, $S(t)Bu\in D(A)$. There exists a positive function $\gamma(\cdot)\in L_1([0,\vartheta];R)$ such that

$$\|AS(t)B\|_{L(U_2,E)}\leqslant\gamma(t)\text{ for a.e. }t\in(0,\vartheta)$$

Let

$$U=U_1\times U_2,\quad u=(u_1,u_2),\quad \Lambda\in L(U,E)$$
$$\Lambda u=\Lambda(u_1,u_2)=\Lambda_1u_1+\Lambda_2u_2,\quad \forall u_1\in U_1$$
$$u_2\in U_2,\quad \Lambda_1=\Pi^{-1}B_1,\quad \Lambda_2=\Pi^{-1}(\sigma B-\lambda_0B)-B,\quad \Pi=$$
$$A-\lambda_0I,\quad \lambda_0\in\rho(A)$$

where $\rho(A)$ is the resolvent of A. Then system (1.3) can be rewritten in the form

$$y^{\cdot}=Az+B_1u_1+\sigma Bu_2+f,\quad y=z+Bu_2,\quad 0\leqslant t\leqslant\vartheta \tag{3.1}$$

and this system in turn can be transformed into an equivalent abstract Cauchy problem for the system /8/

$$w^{\cdot}=Aw+\Lambda_1u_1+\Lambda_2u_2+\Pi^{-1}f,\quad y=\Pi w \tag{3.2}$$

A weak solution of system (3.2), hence also of systems (3.1), (1.3), is a function $y(\cdot)=y(\cdot;y_0,u(\cdot))\in C([0,\vartheta];E)$ defined by

$$y(t)=V(t;y_0,u(\cdot))=S(t)y_0+\int_0^t\Pi S(t-s)(\Lambda u(s)+\Pi^{-1}f(s))\,ds$$

Under the above conditions, there is a unique weak solution for any $y_0\in E$ and $u(\cdot)\in L_\infty([0,\vartheta];U)$ /8/.

To compute the control $u_*(\cdot)\in U_*$, one can use the algorithm described in Sect.2. This is done with the following system playing the part of the model M:

$$p^{\cdot}(t)=Ap(t)+\Lambda v^h(t)+\Pi^{-1}f(t),\quad z(t)=\Pi p(t),\quad 0\leqslant t\leqslant\vartheta \tag{3.3}$$

where we mean by the motion $z(t)=V(t;\psi_0,v^h(\cdot))$. The mappings $\alpha(h)$ and $\delta(h)$ are determined by the conditions

$$\alpha(h)\to 0,\quad h\alpha^{-1}(h)\to 0,\quad \delta(h)\alpha^{-1}(h)\to 0\quad\text{as}\quad h\to 0+$$

The control $v^h(\cdot)$ in the model must be evaluated according to the rule (2.4), assuming that

$$v_i = v_i\,(t) \quad \text{for a.e. } t \in [\tau_i, \tau_{i+1})$$
$$l_1\,(s_i^*, v_i\,(\cdot)) \leqslant l_1\,(s_i^*) + h\delta_i, \quad s_i^* = S\,(\delta_i)\,\Pi^{-1}\,(z\,(\tau_i) - \psi_i)$$
$$l_1\,(s_i^*, v_i\,(\cdot)) = 2\left(s_i^*, \int\limits_0^{\delta_i} S\,(\delta_i - s)\,\Lambda v_i\,(\tau_i + s)\,ds\right)_E$$
$$\alpha \int\limits_0^{\delta_i} \| v_i\,(\tau_i + s)\|^2\,ds, \quad l_1\,(s_i^*) = \inf\,\{l_1\,(s_i^*;$$
$$v\,(\cdot)) \mid v\,(t) \in P \ \text{for a.e. } t \in [\tau_i, \tau_{i+1}]\}$$

With this choice of the model, the maps $\alpha\,(h)$ and $\delta\,(h)$ and the rule for computing the control $v^h\,(\cdot)$, Theorem 2.1 holds if $z_0 = \psi_0$.

Remarks. 4. Let

$$\omega_1\,(\delta) = \sup\,\{|\,\{S\,(t) - S\,(\xi)\}\,\Lambda\, v\,| \mid t, \xi \in [0, \delta], v \in P\}$$
$$\omega_1\,(\delta) \to 0 \ \text{as} \ \delta \to 0$$
$$\{\omega_1\,(\delta\,(h)) + \delta\,(h)\}\,\alpha^{-1}\,(h) \to 0 \quad \text{as} \quad h \to 0+.$$

Then one can define $v^h\,(\cdot)$ via (2.4), (2.5), assuming in (2.5) that $B = \Lambda, s_i = s_i^*$.

5. Theorem 2.1 is also true if the semigroup $\{S\,(t) \mid t \geqslant 0\}$ is ω-dissipative: $|\,S\,(t)\,z\,| \leqslant e^{\omega t}\,|\,z\,|$, $\forall z \in E$.

4. *Examples*. 1. Let us consider the boundary control problem for a linear parabolic system with Dirichlet conditions. Let $\Omega \subset R^n$ be a bounded open set with a sufficiently smooth boundary and

$$y_t^{\cdot}\,(\eta, t) - \Delta y\,(\eta, t) = f\,(\eta, t) \ \text{in} \ \Omega \tag{4.1}$$
$$y\,(\eta, t)\,|_\Gamma = u\,(\eta, t) \ \text{for} \ t \in (0, \vartheta),\ y\,(\eta, 0) = y_0\,(\eta)$$
$$y_0\,(\eta) \in L_2\,(\Omega),\ f \in L_2\,(Q), \quad u \in L_2\,(\Sigma),\ |\,u\,(\eta, t)\,|_{L_2(\Gamma)} \leqslant 1 \ \text{for a.e.}\ t \in (0, \vartheta).$$

In order to rewrite system (4.1) in the form (1.3), we proceed as follows /8/:

$$E = U_1 = L_2\,(\Omega),\ X = H^{-1/2}\,(\Gamma), \quad U_2 = L_2\,(\Gamma), \quad B_1 = 0$$
$$B_2 = I, \quad \sigma = \Delta, \quad D\,(\sigma) = \{y \in L_2\,(\Omega) \mid \Delta y \in L_2\,(\Omega)\}$$

where τ is the trace operator: $\tau y \in H^{-1/2}\,(\Gamma)$ if $y \in D\,(\sigma)$, $A = \Delta$, $D\,(A) = H_0^1\,(\Omega) \cap H^2\,(\Omega)$. The continuous linear operator B: $L_2\,(\Gamma) \to L_2\,(\Omega)$ is defined by $Bu = w_u$, where $w_u \in L_2\,(\Omega)$ is the unique generalized solution of the Dirichlet problem $\Delta w_u = 0$ on Ω, $w_u\,|_\Gamma = 0$, i.e.,

$$\int\limits_\Omega w_u \Delta\psi d\eta = \int\limits_\Gamma u\,\frac{\partial\psi}{\partial n}\,ds \ \forall \psi \in H_0^1\,(\Omega) \cap H^2\,(\Omega)$$

Conditions 1-4 of Sect.3 are satisfied, with $\gamma = Ct^{-3/4}$ /8/. The phase state of the system at $\lambda_0 = 0$ is found from the formula

$$z\,(t) = S\,(t)\,y_0 - \int\limits_0^t \Delta S\,(t - s)\,Bu\,(s)\,ds$$

and the control $v_i\,(t), t \in [\tau_i, \tau_{i+1})$, at time τ_i is determined by the conditions

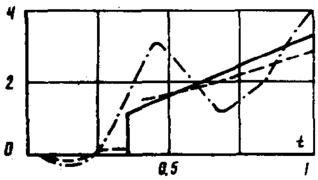
$$v_i\,(t) = v_*\,(t - \tau_i) \ \text{for a.e.}\ t \in [\tau_i, \tau_{i+1})$$
$$2\int\limits_0^{\delta_i} \left(\frac{\partial}{\partial n}\,\Delta^{-1} S\,(\delta_i - s)\,s_i^*\,|_\Gamma, v_*\,(s)\right)_{L_2(\Gamma)} + \alpha \int\limits_0^{\delta_i} \|v_*\,(s)\|^2_{L_2(\Gamma)}\,ds \leqslant$$
$$\inf\left\{2\int\limits_0^{\delta_i}\left(\frac{\partial}{\partial n}\,\Delta^{-1} S\,(\delta_i - s)\,s_i^*\,|_\Gamma, \quad v\,(s)\right)_{L_2(\Gamma)} ds +\right.$$
$$\left.\alpha \int\limits_0^{\delta_i} \|v\,(s)\|^2_{L_2(\Gamma)}\,ds \mid \|v\,(s)\|_{L_2(\Gamma)} \leqslant 1 \ \text{for a.e.}\ s \in [0, \delta_i]\right\} + h\delta_i$$
$$s_i^* = S\,(\delta_i)\,A^{-1}\,(z_i - \psi_i), \quad |\,z_i^* - z\,(\tau_i)\,|_{L_2(\Omega)} \leqslant h$$

2. The problem of reconstructing u in system (1.1) was simulated on a computer with the following data:

$$n = 2,\quad \Omega = \{(\eta_1, \eta_2) \mid 0 < \eta_1 < 1,\quad 0 < \eta_2 < 1\}$$
$$T = [0, 1],\quad y_0(\eta) = 0,\quad P = \{u(\eta) \in L_2(\Omega) \mid$$
$$|u(\eta)| \leqslant 20 \text{ for a.e. } \eta \in \Omega\}$$
$$y(t, \eta) = \begin{cases} 5t(1-\eta_1)(1-\eta_2)\eta_1\eta_2, & \eta_1 \leqslant t \\ 0, & \eta_1 > t \end{cases}$$
$$u_*(t, \eta) = \begin{cases} y_t{}'(t, \eta) - \Delta y(t, \eta), & \eta_1 \leqslant t \\ 0, & \eta_1 > t \end{cases}$$

The phase trajectory of the model $z(\cdot)$ was computed by an explicit grid method (/11/. Chap.6) with time step-size δ. The region Ω was divided into squares of side $h_1 = 0.05$ and replaced by a uniform grid of mesh h_1. The construction of the control $v^h(\cdot)$ in the model used only the grid values of $\psi(\tau_i, \eta)$.

The figure shows sections at the grid-point $\eta_1 = 0.27, \eta_2 = 0.27$ of the control $u_*(t, \eta)$ (solid curve), as well as values of $v^h(\eta, t)$ determined for $\delta = 1/1200, \psi(\eta, \tau_i) = y(\eta, \tau_i)$ (dashed curve) and $\delta = 1/1500, \psi(\eta, \tau_i) = y(\eta, \tau_i) + 0.25\sin(-10t)$ (dash-dotted curve). As shown by a numerical experiment, the quantity $\|u_*(\cdot) - v^h(\cdot)\|_{L_2(Q)}$ is equal to 0.82989 in the first case and 4.66924 in the second.



Remarks. 6. In a computer simulation of controls when E is a Sobolev space on Ω, it is natural to replace Ω by a certain grid $\bar{\omega} = \{\eta_j \mid j \in [1: N]\} \subset \Omega$ of mesh h_1 and to assume that the values of $\psi(\eta, \tau_i)$ are measured at the grid-points η_j. As the equation of the model one can then take the difference analogue of Eq.(2.3) or (3.3), with $v^h(\cdot)$ replaced by grid functions $v^h(\eta_j, \tau_i)$.

Remarks. 7. For systems with distributed parameters, described by simple boundary-value problems for equations of parabolic and hyperbolic type, questions analogous to those considered here were discussed, in particular, in a lecture by Yu.S. Osipov, entitled "Control and modelling in multidimensional systems", delivered at the general meeting of the Department of Mechanics and Control Processes in November 1984.

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